1. Show that a group that has only a finite number of subgroups must be a finite group.

2. Let r and s be positive integers. Show that $\{nr + ms \mid n, m \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}$.

3. Consider S_n for a fixed $n \geq 2$ and let σ be a fixed odd permutation. Show that every odd permutation in S_n is a product of σ and some permutation in A_n.

4. Let G be a group of order pq, where p and q are prime numbers. Show that every proper subgroup of G is cyclic.

5. Show that a finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

6. Let G be a group, and let $g \in G$. Let $\phi_g : G \to G$ be defined by $\phi_g(x) = gxg^{-1}$ for $x \in G$. For which $g \in G$ is ϕ_g a homomorphism?

7. Compute the indicated quantities for the given homomorphism.

   Ker(ϕ) and $\phi(3, 10)$ for $\phi : \mathbb{Z} \times \mathbb{Z} \to S_{10}$ where $\phi(1, 0) = (3, 5)(2, 4)$ and $\phi(0, 1) = (1, 7)(6, 10, 8, 9)$

8. Let G be a group, h an element of G, and n a positive integer. Let $\phi : \mathbb{Z}_n \to G$ be defined by $\phi(i) = h^i$ for $0 \leq i \leq n$. Give a necessary and sufficient condition (in terms of h and n) for ϕ to be a homomorphism. Prove your assertion.

9. Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G.

10. Show that if H and N are subgroups of a group G, and N is normal in G, then $H \cap N$ is normal in H. Show by an example that $H \cap N$ need not be normal in G.

11. Show that if G is nonabelian, then the factor group $G/Z(G)$ is not cyclic. [*Hint:* Show the equivalent contrapositive, namely, that if $G/Z(G)$ is cyclic then G is abelian (and hence $Z(G) = G$).]

12. **Show that a subset A of a group G cannot be a left coset of two distinct subgroups of G. If A is a left coset of some subgroup of G, then prove that A is also a right coset of some subgroup of G.**